Calculus of multivariable functions

Definition 1. A function $y = \mathrm{f}(x_1, \ldots, x_n)$ is called a *function of* n *independent variables* if there exists one and only one value of y in the range of f for each tuple of real number $(x_1, \ldots, x_n)$ in the domain of f. Here y is called the *dependent variable* while x_i, $i = 1, \ldots, n$, the *independent variables*.

Definition 2. Let a multivariable function be

$$y = \mathrm{f}(x_1, \ldots, x_n)$$

The *partial derivative* of y with respect to x_i, where $1 \leq i \leq n$, is a measure of the instantaneous rate of change of y with respect to x_i while x_j is held constant for all $j \neq i$, where $1 \leq j \leq n$. This partial derivative is defined as

$$\frac{\partial y}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{\mathrm{f}(\ldots, x_i + \Delta x_i, \ldots) - \mathrm{f}(x_1, \ldots, x_n)}{\Delta x_i}$$

and can be written in either one of the following forms.

$$\frac{\partial y}{\partial x_i}, \ \frac{\partial \mathrm{f}}{\partial x_i}, \ \mathrm{f}_{x_i}(x_1, \ldots, x_n), \ \mathrm{f}_{x_i}, \text{ or } y_{x_i}$$

Theorem 1. Let $z = \mathrm{g}(x, y) \cdot \mathrm{h}(x, y)$. Then,

$$\frac{\partial z}{\partial x} = \mathrm{g} \cdot \frac{\partial \mathrm{h}}{\partial x} + \mathrm{h} \cdot \frac{\partial \mathrm{g}}{\partial x}$$

and

$$\frac{\partial z}{\partial y} = \mathrm{g} \cdot \frac{\partial \mathrm{h}}{\partial y} + \mathrm{h} \cdot \frac{\partial \mathrm{g}}{\partial y}$$

Theorem 2. Let $z = [\mathrm{g}(x, y)]^n$. Then,

$$\frac{\partial z}{\partial x} = n\mathrm{g}^{n-1} \cdot \frac{\partial \mathrm{g}}{\partial x}$$

and

$$\frac{\partial z}{\partial y} = n\mathrm{g}^{n-1} \cdot \frac{\partial \mathrm{g}}{\partial y}$$

Definition 3. Let $z = \mathrm{f}(x, y)$. Then, the *second-order direct partial derivatives* are

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

These are also written

$$\mathrm{f}_{xx},\ (\mathrm{f}_x)_x,\ \frac{\partial^2 z}{\partial x^2} \quad \text{and respectively} \quad \mathrm{f}_{yy},\ (\mathrm{f}_y)_y,\ \frac{\partial^2 z}{\partial y^2}$$

The *cross partial derivatives* are

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

These are also written as

$$\mathrm{f}_{xy},\ (\mathrm{f}_x)_y,\ \frac{\partial^2 z}{\partial y \partial x} \quad \text{and respectively} \quad \mathrm{f}_{yx},\ (\mathrm{f}_y)_x,\ \frac{\partial^2 z}{\partial x \partial y}$$

Theorem 3. For a multivariable function $z = \mathrm{f}(x, y)$ to be a *relative maximum* at (a, b) necessarily $\mathrm{f}_x, \mathrm{f}_y = 0$, and $\mathrm{f}_{xx}, \mathrm{f}_{yy} < 0$ and $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ at that point. For the same at the same to be a *relative minimum*, necessarily $\mathrm{f}_x, \mathrm{f}_y = 0$, and $\mathrm{f}_{xx}, \mathrm{f}_{yy} > 0$ and $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ there. Moreover, an *inflection point* is a point (a, b) at which $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < (\mathrm{f}_{xy})^2$, and both f_{xx} and f_{yy} have the same sign. On the other hand, a *saddle point* is a point (a, b) at which $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < (\mathrm{f}_{xy})^2$, but f_{xx} and f_{yy} are of different signs.

Procedure 1 *Procedure for determining a critical point of a function with two independent variables*

Given $z = \mathrm{f}(x, y)$ and a point (a, b)**, at this point,**

if $\mathrm{f}_x = 0$ and $\mathrm{f}_y = 0$ **then**

  (a, b) is a critical point

  if $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ **then**

    if $\mathrm{f}_{xx} < 0$ and $\mathrm{f}_{yy} < 0$ **then**

      (a, b) is a relative maximum of z

    elseif $\mathrm{f}_{xx} > 0$ and $\mathrm{f}_{yy} > 0$ **then**

      (a, b) is a relative minimum of z

    else †

    endif

elseif $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < (\mathrm{f}_{xy})^2$ **then**
if $\mathrm{f}_{xx} \cdot \mathrm{f}yy > 0$ **then**
(a, b) is an inflection point
elseif $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < 0$ **then**
(a, b) is a saddle point
else ‡
endif
else
test inconclusive
endif
else
(a, b) is no critical point
endif

Problem 1. There are two dead ends in Procedure 1. The first one (†) is the case where $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ and either $(\mathrm{f}_{xx} = 0, \mathrm{f}_{yy} = 0)$, $(\mathrm{f}_{xx} = 0, \mathrm{f}_{yy} < 0)$, $(\mathrm{f}_{xx} = 0, \mathrm{f}_{yy} > 0)$, $(\mathrm{f}_{xx} < 0, \mathrm{f}_{yy} = 0)$, $(\mathrm{f}_{xx} > 0, \mathrm{f}_{yy} = 0)$, $(\mathrm{f}_{xx} < 0, \mathrm{f}_{yy} > 0)$, or $(\mathrm{f}_{xx} > 0, \mathrm{f}_{yy} < 0)$. The second one (‡) is where $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} = 0$. Find out what happen in these cases, and thus complete the missing lines of logic in Procedure 1.

Definition 4. By *derivative* $\frac{\mathrm{d}y}{\mathrm{d}x}$ we mean an infinitesimally small change in y with respect to an infinitesimally small change in x. By *differential* $\mathrm{d}y$ and $\mathrm{d}x$ we mean an infinitesimally small change in the values of y and respectively x.

Example 1. For a function of one variable $y = \mathrm{f}(x)$, the total derivative is

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

and the differential of y is

$$\mathrm{d}y = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x$$

For a function of two variables $z = \mathrm{f}(x, y)$ partial derivatives are, the first-order partial derivatives

$$\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y}$$

and the second-order partial derivatives

$$\frac{\partial^2 z}{\partial x^2} \equiv z_{xx},\ \frac{\partial^2 z}{\partial y^2} \equiv z_{yy},\ \frac{\partial^2}{\partial y \partial x} \equiv z_{xy} \text{ and } \frac{\partial^2 z}{\partial x \partial y} \equiv z_{yx}$$

The total differential of z is

$$\mathrm{d}z = \left(\frac{\partial \mathrm{f}}{\partial x}\right) \mathrm{d}x + \left(\frac{\partial \mathrm{f}}{\partial y}\right) \mathrm{d}y$$

and for small changes which are not infinitesimal, $\mathrm{d}x$ becomes Δx and the incremental change formula is

$$\Delta z \approx \left(\frac{\partial \mathrm{f}}{\partial x}\right) \Delta x + \left(\frac{\partial \mathrm{f}}{\partial y}\right) \Delta y$$

Definition 5. The *general production function* is $q = \mathrm{f}(l, k)$, where q is output of the production, l labour and k capital. The *Cobb-Douglas production function* in its general form is

$$q = al^{\alpha}k^{\beta} \tag{1}$$

where a is a constant and $0 < \alpha < 1$, $0 < \beta < 1$, $l > 0$ and $k > 0$.

Example 2. With the Cobb-Douglas production function, the *marginal product of labour* is,

$$p_{lm} = q_l = \frac{\partial q}{\partial l} = a\alpha l^{\alpha-1}k^{\beta} \tag{2}$$

and the *marginal product of capital*

$$p_{km} = q_k = \frac{\partial q}{\partial k} = a\beta l^{\alpha}k^{\beta-1} \tag{3}$$

From this we see that $p_{lm} > 0$ and $p_{km} > 0$.

Theorem 4. From the Cobb-Douglas production function we have the *law of diminishing returns to labour*, which states that $q_{ll} < 0$.

Proof. From Equation 1 in Definition 5,

$$q_{ll} = \frac{\partial^2 q}{\partial l^2} = \frac{\partial}{\partial l}\left(\frac{\partial q}{\partial l}\right) = \frac{\partial p_{lm}}{\partial l} = (\alpha - 1)\frac{\alpha q}{l^2}$$

Since $0 < \alpha < 1$, therefore $q_{ll} < 0$.

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Example 3. Using the Cobb-Douglas production function,

$$q_{kl} = q_{lk} = a\alpha\beta l^{\alpha-1}k^{\beta-1}$$

Therefore, $q_{lk} > 0$ and $q_{kl} > 0$. In other words, p_{lm} increases as capital input k increases, and respectively p_{km} increases as labour input l increases.

Example 4. For the Cobb-Douglas production function in Equation 1 the average product of labour is

$$p_{la} = \frac{q}{l} = al^{\alpha-1}k^{\beta} \tag{4}$$

and the average product of capital is

$$p_{ka} = \frac{q}{k} = al^{\alpha}k^{\beta-1} \tag{5}$$

Example 5. Again using the Cobb-Douglas production function of Equation 1, the marginal product of labour is

$$p_{lm} = \frac{\partial q}{\partial l} = a\alpha l^{\alpha-1}k^{\beta} \tag{6}$$

and the marginal product of capital is

$$p_{km} = \frac{\partial q}{\partial k} = a\beta l^{\alpha}k^{\beta-1} \tag{7}$$

Example 6. From the APL equation, Equation 4, and the MPL equation, Equation 6, and since $0 < \alpha < 1$, therefore $p_{ml} < p_{la}$. Similarly from the APK equation, Equation 5, and the MPK equation, Equation 7, since $0 < \beta < 1$, we have $p_{km} < p_{ka}$.

Example 7. A producer likes to have a positive marginal function, which means that the productivity increases as the input increases. But the second derivative is negative, which means that this rate of increase slows down as time goes by. In practice, the conditions for using labour are,

$$p_{lm} = \frac{\partial q}{\partial l} > 0,\ \frac{\mathrm{d}p_{lm}}{\mathrm{d}l} = \frac{\partial^2 q}{\partial l^2} < 0,\ \text{and}\, p_{lm} < p_{la} \tag{8}$$

The conditions for using capital are similarly,

$$p_{km} = \frac{\partial q}{\partial k} > 0,\ \frac{\mathrm{d}p_{km}}{\mathrm{d}k} = \frac{\partial^2 q}{\partial k^2} < 0,\ \text{and}\, p_{km} < p_{ka} \tag{9}$$

Definition 6. An *isoquant* is a graph in two dimensions, $k = \mathrm{f}(l)$, plotted to represent a production function $q = \mathrm{f}(l, k)$. The slope

$$\frac{\mathrm{d}k}{\mathrm{d}l}$$

is called the *marginal rate of technical substitution.* The value of this slope at (l_0, k_0) is denoted by

$$\left.\frac{\mathrm{d}k}{\mathrm{d}l}\right|_{l_0 k_0}$$

Theorem 5. The slope of an isoquant is the ratio of the marginal products.

Proof. The total differential of $q = \mathrm{f}(l, k)$ is

$$\mathrm{d}q = \left(\frac{\partial q}{\partial l}\right)\mathrm{d}l + \left(\frac{\partial q}{\partial k}\right)\mathrm{d}k$$

Along any isoquant, $\mathrm{d}q = 0$, therefore,

$$0 = \left(\frac{\partial q}{\partial l}\right)\mathrm{d}l + \left(\frac{\partial q}{\partial k}\right)\mathrm{d}k \tag{10}$$

This directly yield, after some manipulation,

$$\frac{\mathrm{d}k}{\mathrm{d}l} = -\frac{q_l}{q_k}$$

Or, from Equation 10 together with Equation's 6 and 7, it follows that,

$$\frac{\mathrm{d}k}{\mathrm{d}l} = -\frac{p_{lm}}{p_{km}}$$

Definition 7. In the Cobb-Douglas production function equation, Equation 1, let both inputs l and k change by the same proportion, and let λ be the constant of this proportionality. Then $q_2 = a(\lambda l)^{\alpha}(\lambda k)^{\beta}$, which leads to $q_2 = \lambda^{\alpha+\beta} q_1$. When $\alpha+\beta = 1$, the case is described as *constant returns to scale*, when $\alpha+\beta < 1$ as *decreasing returns to scale*, and when $\alpha+\beta > 1$ as *increasing returns to scale*.

Definition 8. The *homogeneous* Cobb-Douglas production function of order r is,

$$\mathrm{f}(\lambda l, \lambda k) = \lambda^r \mathrm{f}(l, k)$$

where $r = (\alpha, \beta)$.

Definition 9. A *utility function* expresses utility as a function of goods consumed. In its general form this is,

$$u = \mathrm{f}(x, y)$$

where x and y are the quantities of goods X and respectively Y consumed.

Definition 10. The *Cobb-Douglas utility function* is in its general form,

$$u = ax^{\alpha}y^{\beta}$$

where a is a constant, and $0 < \alpha < 1$, $0 < \beta < 1$, $x > 0$ and $y > 0$.

Definition 11. The *marginal utility* for a utility function with one variable, $u = \mathrm{f}(x)$, is $\frac{\mathrm{d}u}{\mathrm{d}x} = u_x = u_{xm}$. The marginal utility for a utility function with two variables, $u = \mathrm{f}(x, y)$, is $\frac{\partial u}{\partial x} = u_x = u_{xm}$ and $\frac{\partial u}{\partial y} = u_y = u_{ym}$.

Definition 12. The *indifference curve* is a graph $y = \mathrm{f}(x)$ drawn to represent a utility function $u = \mathrm{f}(x, y)$. Its slope $\frac{\mathrm{d}y}{\mathrm{d}x}$ is called the *marginal rate of substitution*. Setting the total differential equal to zero,

$$0 = \mathrm{d}u = \left(\frac{\partial u}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial u}{\partial y}\right)\mathrm{d}y$$

we find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{u_x}{u_y}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{u_{xm}}{u_{ym}}$$

Definition 13. Let a demand function be

$$q_a = \mathrm{f}(p_a, y, p_b) \tag{11}$$

where q_a is the quantity demanded of good a, p_a the price of a, y consumer's income, and p_b the price of another good b. Then, the *price elasticity of demand* is,

$$\varepsilon_d = \frac{\partial q_a}{\partial p_a}\frac{p_a}{q_a}$$

The *income elasticity of demand* is,

$$\varepsilon_y = \frac{\partial q_a}{\partial y}\frac{y}{q_a}$$

And the *cross-price elasticity of demand* is,

$$\varepsilon_c = \frac{\partial q_a}{\partial p_b}\frac{p_b}{q_a}$$

Example 8. With the demand function as in Equation 11, the partial elasticity with respect to labour is,

$$\varepsilon_{ql} = \frac{\partial q}{\partial l}\frac{l}{q}$$

And from Equation's 6 and 4, this leads to,

$$\varepsilon_{ql} = \frac{p_{lm}}{p_{la}}$$

For the Cobb-Douglas production function, Equation 1, then $\varepsilon_{ql} = \alpha$.

Example 9. Again, with the demand function as in Equation 11, the partial elasticity with respect to capital is,

$$\varepsilon_{qk} = \frac{\partial q}{\partial k}\frac{k}{q}$$

Then, from Equation's 7 and 5,

$$\varepsilon_{qk} = \frac{p_{km}}{p_{ka}}$$

For the Cobb-Douglas production function, Equation 1, we have $\varepsilon_{qk} = \beta$.